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1 Numerical Methods in Stochastic Control

The second edition of our book [4] on numerical methods in stochastic control has appeared. The book and the methods contained therein are now the standard in the field. It contains the most comprehensive development of numerical algorithms and associated convergence proofs for a large part of the current forms of stochastic control problems in continuous time. The PI's algorithms (and proof techniques) are the algorithms of choice for the bulk of continuous time stochastic control problems. In addition to the broad coverage of the first edition, it gives numerical algorithms and proofs for problems where the variance term is controlled, and for jump–diffusions where the jump is controlled. Important applications of jump control occur, for example, in communications theory. Consider, for example, a system where a server divides its time between several queues whose input processes are bursty, and the individual connections are subject to random breakdown or fading. The control problem is the scheduling of the server and this must be done continuously. A jump increase in the total system workload can occur when some connection breaks down or fades and the work in the available queues is less than the server can handle, but customers continue to arrive at the unavailable queues, so there is undesired idle time. The control policy affects the jump sizes. Traditional methods cannot handle such problems. The standard use of the Poisson measure driven model is no longer adequate, and a general theory is developed. Additionally, the book contains a thorough development of deterministic problems that arise in control and in the calculus of variations, and includes discontinuous or unbounded dynamical terms, with applications to image reconstruction, large deviations, and elsewhere. The algorithms are about the fastest and most stable available, and there are convergence proofs for all of them.

Numerical methods in stochastic control are now a fundamental tool for the solution and investigation of stochastic problems, whether controlled or not. In any particular application, one is not usually interested in a single cost criterion, whether it is the mean number in the system, the mean waiting time or anything else. There are usually several conflicting criteria, where improving one might mean hurting another. If a small improvement in one comes at the expense of a large loss in another, then it will be unacceptable. But whether or not any particular weighting of the criteria will yield an optimal policy that is an acceptable solution (in practice) is not usually known before a problem is solved. A very useful role of optimization is to explore the possible tradeoffs; what happens under the optimal controls for various given sets of weights. One would numerically solve a sequence of limit problems to get controls that are optimal under different weighings of the basic criteria of interest, to get the general structure and the parametric dependencies of the controls and costs. The resulting quantitative and qualitative information and systematic exploration of the possible tradeoffs among the various cost components can be extremely useful in design, as seen in the study of a communications system in [5, 6]. This approach represents a significant application of our numerical methods and places great demands on them. The method is robust and simplifies the analysis (both analytical and numerical).

Numerical Approximations for Stochastic Differential Games: The Ergodic Case. The Markov chain approximation method is a widely used, relatively easy to use, and efficient family of methods for the bulk of stochastic control problems in continuous time, for reflected-jump-diffusion type models. It has been shown to converge under broad conditions, and there are good algorithms for solving the numerical problems, if the dimension is not too high. In [3] we consider a class of stochastic differential games with a reflected diffusion system model and ergodic cost criterion and where the controls for the two players are separated in the dynamics and cost function. It is shown that the value of the game exists and that the numerical method converges to this value as the discretization parameter goes to zero. The actual numerical method solves a stochastic game for a finite state Markov chain and ergodic cost criterion. The essential conditions are nondegeneracy and that a weak local consistency condition hold "almost everywhere" for the numerical approximations, just as for the control problem.

Heavy Traffic Analysis of Controlled Queueing and Communication Networks. Another major achievement was the appearance of this new book [2]. It is, by far, the most comprehensive

on the subject. It provides a thorough development of the powerful methods of heavy traffic analysis and approximations with applications to a wide variety of stochastic (e.g., queueing and communication) networks, for both controlled and uncontrolled systems. The approximating models are reflected stochastic differential equations. The analytical and numerical methods yield considerable simplifications and insights and good approximations to both path properties and optimal controls under broad conditions on the data and structure. The general theory is developed, with possibly state dependent parameters, and specialized to many different cases of practical interest. Control problems in telecommunications and applications to scheduling, admissions control, polling, and elsewhere are treated. There is a detailed survey of reflected stochastic differential equations, weak convergence theory, methods for characterizing limit processes, and ergodic problems.

Stability and Control of Mobile Communications Systems With Time Varying Channels. Consider the forward link of a mobile communications system with a single transmitter and rather arbitrary randomly time varying channels connecting the base to the mobiles. Data arrives at the base in some random way (and might have a bursty character) and is queued according to the destination until transmitted. The main issues are the allocation of transmitter power and time to the various queues in a queue- and channel-state dependent way to assure stability and good operation. The control decisions are made at the beginning of the (small) scheduling intervals. Stability methods are used in [1] to allocate time and power. Many schemes of current interest can be handled: For example, CDMA with control over the bit interval and power per bit, TDMA with control over the time allocated, power per bit, and bit interval, as well as arbitrary combinations. There might be random errors in transmission which require retransmission. The channel-state process might be known or only partially known. The details of the scheme are not directly involved; all essential factors are incorporated into a "rate" and "error" function. The system and channel process are scaled by speed. Under a stability assumption on a model obtained from the "mean drift," and some other natural conditions, it is shown that the scaled physical system can be controlled to be stable, uniformly in the speed, for fast enough speeds. Owing to the non-Markov nature of the problem, we use the perturbed Liapunov function method, which is very useful for the analysis of non-Markovian systems. Finally, the stability method is used to actually choose the power and time allocations. The allocation will depend on the Liapunov function. But each such function corresponds loosely to an optimization problem for some performance criterion. Since there is a choice of Liapunov functions, various performance criteria can be taken into account in the allocations. The resulting controls are quite reasonable. The power of the method is due to the rather general conditions under which it works and the reasonableness of the controls.

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Large Deviation Approximations for Occupancy Problems. The paper [1] (with C. Nuzman and P. Whiting of Bell Labs) was completed. In the occupancy problem one considers the distribution of r balls in n cells, with each ball assigned independently to a given cell with probability $1/n$. In the paper just mentioned a large deviation approximation as r and n tend to infinity was proved. Occupancy problems have many applications in computer science, mathematical biology and elsewhere, but the original motivation for this work is the design of an optical communication switch, where blocking probabilities are of central importance. Here the urns correspond to channels and the balls to packets being routed through the switch. In order to analyze the problem a dynamical model is first considered, where the balls are placed in the cells sequentially and "time" corresponds to the number of balls that have already been thrown. A complete large deviation analysis of this "process level" problem is carried out, and the rate function for the original problem is then obtained via the contraction principle. The variational problem that characterizes this rate function is analyzed, and (in sharp contrast to most analyses of large deviations for Markov processes), a complete and explicit solution is obtained. The minimizing trajectories and minimal cost are identified up to two constants, and the constants are characterized as the unique solution to an elementary fixed point problem. These results are then used to solve a number of interesting problems, including the overflow problem and the so-called partial coupon collector's problem.

Regulation and Analysis of Stochastic Networks. Research in this area proceeded along several directions. Key issues in our investigations included: (i) the development of approximate models that allow for explicit (or nearly explicit) construction of the optimal routing/service policies, and (ii) robustness and the ability to deal with model perturbations. The problems considered encompass a number of different formulations, each of which emphasizes different qualitative properties of the resulting controlled network. These include risk-sensitive control and the control of rare events, optimal control of "fluid" models, and optimally robust control of such models. "Fluid" models are approximate models that are obtained under a law of large numbers scaling. In all cases the system model is constrained, and in most problems this is a dominant feature. To handle the constraints, we model the state dynamics of the approximate (or limit) models in terms of an appropriate Skorokhod Problem and corresponding constrained ordinary or stochastic differential equations. The different problem setups (e.g., control of rare events versus control of fluid models) lead to problems of the same basic form, which is a Skorokhod Problems with (relatively) simple dynamics and simple cost structures.

Paper [2] (joint with K. Ramanan of Bell Labs) deals with a controlled constrained ordinary differential equation, and with a cost that depends only on the control. Such problems occur in the (ordinary) control of fluid models, such as control of a network with the objective of reducing backlogs in minimum time. Our analysis gives an explicit finite-dimensional representation of the value function, and identifies all optimal controls. Reference [3] (also with Ramanan) considers the problem of characterizing the manner in which rare events occur in networks under heavy traffic. It shows how time reversal arguments can be applied to rewrite the control problem defined by a straightforward large deviations analysis into the form of [2], and then shows how explicit finite-dimensional solutions can be obtained for some interesting classes of problems (in arbitrary dimension). A three dimensional example is worked out in detail for illustrative purposes.

The paper [4] considers the robust optimal control of a law of large numbers approximation of a stochastic network. The robust control problem is formulated as a dynamic differential game, with one player choosing the policies that determine service and routing assignments, and the other choosing quantities such as the arrival and service rates, subject to constraints. The cost to be minimized by the first player and maximized by the second is the time till the origin is reached. The robust formulation allows one to differentiate between the many policies (at the fluid level) that are optimal for an ordinary cost. An explicit formula is given for the value function, and some of its

basic properties are studied. The problem of policy synthesis for particular classes of problems has also been studied, and for these cases we have explicitly constructed a robustly optimal control.

A complementary regulation problem is considered in [5], joint with R. Atar and A. Shwartz of The Technion. This paper considers the problem of robust and risk-sensitive control of a stochastic network. In controlling such a network, an escape time criteria (rather than time to reach the origin) is useful if one wishes to regulate the occurrence of large buffers and buffer overflow. A risk-sensitive escape time criteria is formulated, which in comparison to the ordinary escape time criteria penalizes exits which occur on short time intervals more heavily. The properties of the risk-sensitive problem are studied in the large buffer limit, and related to the value of a deterministic differential game with constrained dynamics. We prove that the game has value, and explicit solutions are obtained to illustrate how the results may be applied.

Approximations and analysis of jump processes for optimal stopping. The papers [6] and [7] are both joint with H. Wang of Brown. The first considers a basic issue in most problems of optimal stopping. The optimal policy is usually obtained from a continuous time model, since it is only in this case that an explicit solution to the corresponding variational problem may be found. In practice, however, one may be restricted to stop only at a fixed set of discrete times. In this paper we identified the rates of convergence of both the optimal costs and the stopping regions, and provided simple formulas for the rate coefficients. The second paper considered optimal stopping problems where the ability to stop depends on exogenous Poisson signal process – one can only stop at the Poisson jump times. Even though the time variable in these problems has a discrete aspect, a variational inequality can be obtained by considering an underlying continuous time structure. We derived the asymptotic behavior of the value functions and optimal exercise boundaries as the intensity of the Poisson process went to infinity, or, roughly speaking, as the problems converge to the classical continuous-time optimal stopping problems.

[1] Large deviation asymptotics for occupancy problems (with Carl Nuzman and Phil Whiting), submitted to *Annals of Probability*.

[2] An explicit formula for the solution of certain optimal control problems on domains with corners (with K. Ramanan), an invited paper in the special issue of *Probab. Th. and Math. Stat.* dedicated to A. Skorokhod **63**, (2000), pp. 32–48.

[3] A time reversed representation for the tail probabilities of stationary reflected Brownian motion (with K. Ramanan), to appear in *Stoch. Proc. and Their Appl.*

[4] Explicit solution to a robust queueing control problem, submitted to *SIAM J. on Control and Opt.*

[5] An escape time criterion for queueing networks: Asymptotic risk-sensitive control via differential games (with Rami Atar and Adam Shwartz), submitted to *Math. of OR*.

[6] On the convergence from discrete to continuous time in an optimal stopping problem (with H. Wang), submitted to *Annals of Applied Probability*.

[7] Optimal stopping with random intervention times (with H. Wang), *Adv. Applied Probability*, **34**, (2002), 1–17.